INTEGRODIFFERENTIAL AND INTEGRAL EQUATIONS OF EQUILIBRIUM OF THIN ELASTIC SHELLS

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In the following we present a method based on application of the work reciprocity theorem for deriving integrodifferential and integral equations of equilibrium of thin elastic shells in terms of displacements. This is a modification of the well-known Somiglian method [5], from which it differs by a special choice of the system of the "auxiliary" displacements; this choice permits of representing the displacement of a point of the middle-surface of the shell by the sum of two displacements: the displacement of the corresponding point of the middle plane of a plate - the "map" of the middle surface of the shell - and a supplementary displacement depending in particular on the curvature of the middle surface of the shell. In the case of cylindrical shells the method was used by the author over ten years ago [4c].

In the present paper it is extended to the treatment of shells of any form of the middle surface.

1. Integral Equation of Equilibrium of a Circular Arch. Consider the simplest example of application of our method to the solution of a one-dimensional problem. This example will permit of clarifying certain particular points of the method before extending it to the treatment of the two-dimensional problems of equilibrium of elastic shells.

The example deals with the state of equilibrium of a circular arch under the action of a concentrated unit force $\mathbf{P}$, directed along a normal to the arch (Fig. 1). We consider the latter as a thin rod rigidly fixed at its ends. The length of the undeformed rod AB will be denoted by l. The position of a point of the rod will be defined by the arch coordinate s, measured from the point A. The arch coordinate of the point of application of the force will be denoted by η. The points of the arch will be related to the points of a beam A_1B_1 of length l, which is the "map" of the arch (Fig. 2). The position of a point on the undeformed rod will be defined by the coordinate s, measured from the point A_1. The scale of the arch coordinates is the same on the curve AB and on the straight line A_1B_1.

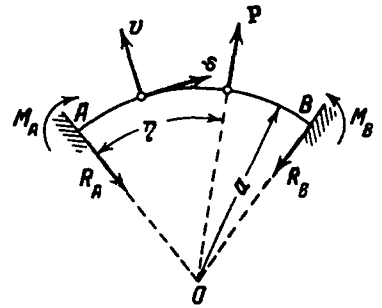

Fig. 1.

We assume that the beam is acted upon by a concentrated unit force **P**, applied at a point with the arch coordinate ξ and directed along the normal to the undeformed axis A_1B_1 of the beam; the ends of the latter are simply supported. Let us derive, by means of the work reciprocity theorem, the interdependence between the radial displacements $u(s, \eta)$ of the circular arch, and the deflections $y(s, \xi)$ of the beam, assuming that the rigidity EI is constant in both arch and beam and that this constant is the same in both. Displacements tangential to the arch will be disregarded.

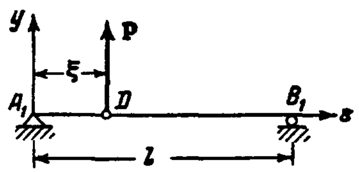

Fig. 2.

From the elementary theory of beams in bending we find

$$y(s, \xi) = \frac{s(l-\xi)}{6EIl}(2\xi l - \xi^2 - s^2) \qquad (s < \xi) \tag{1.1.1}$$

$$y(s, \xi) = \frac{\xi(l-s)}{6EIl}(2sl - \xi^2 - s^2) \qquad (s > \xi) \tag{1.1.2}$$

The rotation angles of the beam at its ends are defined by

$$\varphi(0, \xi) = \frac{\xi(l-\xi)(2l-\xi)}{6EIl}, \qquad \varphi(l, \xi) = -\frac{\xi(l-\xi)(l+\xi)}{6EIl} \tag{1.2}$$

Let us consider $y(s, \xi)$ as the radial displacement of a point of the circular arch. In correspondence herewith the rotation angles $\theta(0, \xi)$ and $\theta(l, \xi)$ of the ends of the arch are determined by the formulas (1.2). In the case of the arch, the displacement $y(s, \xi)$ is produced by the action of a concentrated unit force in conjunction with that of some additional radially distributed load, according to the curvature of the arch. This additional load can be determined by means of the theory of thin rods from the equation of equilibrium [5]

$$EI\left(\frac{d^4y}{ds^4} + \frac{1}{a^2}\frac{d^2y}{ds^2}\right) = q \tag{1.3}$$

where q represents the radial load distributed along the circular arch.

From (1.1.1) and (1.1.2) we find

$$q(s, \xi) = \begin{cases} -\dfrac{1}{a^2 l} s(l-\xi) & (s < \xi) \\ -\dfrac{1}{a^2 l} \xi(l-s) & (s > \xi) \end{cases} \qquad q(s, \xi) = q(\xi, s) \tag{1.4}$$

Applying the work reciprocity theorem we consider, as the first system of loads on the arch, the concentrated unit force **P**, acting at the point $C(\eta)$, the reactions $R(0, \eta)$, $R(l, \eta)$ and the reactive moments $M(0, \eta)$, $M(l, \eta)$ produced by that force (Fig. 1). The radial displacements, corresponding to the first system, are $u(s, \eta)$. The second and auxiliary system of loads consists of the concentrated unit force applied at the point $D(\xi)$ of the arch corresponding to the point $D(\xi)$ of the beam, the radially distributed load $q(s, \xi)$ determined by formula (1.3), the reactions $Q(0, \xi)$, $Q(l, \xi)$ and the reactive moments $L(0, \xi)$, $L(l, \xi)$. The radial displacements corresponding to the auxiliary system are $y(s, \xi)$.

For these systems of loads and displacements the work reciprocity theorem gives

$$u(\xi, \eta) = y(\eta, \xi) - \int_0^l q(s, \xi)\, u(s, \eta)\, ds + M(0, \eta)\,\theta(0, \xi) + M(l, \eta)\,\theta(l, \xi) \tag{1.5}$$

Since $M(0, \eta)$ and $M(l, \eta)$ are expressed by derivatives of the displacements $u(s, \eta)$ at the points $s = 0$ and $s = l$, the relation (1.5) is an integrodifferential equation of equilibrium of the arch. The meaning of this equation is evident. The second term of its right-hand member depends, in accordance with (1.3), on the curvature of the undeformed arch, while the third and fourth terms depend on the difference of the boundary conditions of the arch, fixed at its ends, and of the beam, simply supported at its ends. If the fixity conditions of the ends of arch and beam are the same, then, as is easily seen, the terms depending on the difference in the boundary conditions will disappear, and the relation (1.5) then becomes a Fredholm integral equation of the second kind with a regular kernel.

To reduce the relation (1.5) to a Fredhom integral equation of the second kind, it is also possible to start from the following consideration: the displacements $y(s, \xi)$, which have a singularity at the point $D(\xi)$ produced by the action of the concentrated unit force $\mathbf{P}$, can be amplified by arbitrary displacements $y_1(s, \xi)$ continuous with respect to s and having derivatives with respect to s continuous up to the fourth order inclusive. These additional displacements correspond to the action of some additional loads at the ends of the arch, and of an additional load continuously distributed along the length of the arch. Assume the additional auxiliary displacements to fulfil the conditions

$$y(0, \xi) + y_1(0, \xi) = 0, \qquad y(l, \xi) + y_1(l, \xi) = 0 \tag{1.6.1}$$

$$\frac{\partial y(0, \xi)}{\partial s} + \frac{\partial y_1(0, \xi)}{\partial s} = 0, \qquad \frac{\partial y(l, \xi)}{\partial s} + \frac{\partial y_1(l, \xi)}{\partial s} = 0 \tag{1.6.2}$$

i.e. the new auxiliary displacements and the corresponding rotation angles at the ends of the arch are zero.

To fulfil these conditions it is sufficient to assume

$$y_1(s, \xi) = y_1(\xi, s) = -\frac{s\xi(l-s)(l-\xi)}{6EIl^3}[l(s+\xi) - 2(l^2 + s\xi)] \tag{1.7}$$

The additional radial auxiliary load $q_1(s, \xi)$, distributed along the arch and corresponding to the displacement $y_1(s, \xi)$, in accordance with (1.3) is represented by the expression

$$q_1(s, \xi) = \frac{\xi(l-\xi)}{a^2l^3}[2s\xi - l(s+\xi) + l^2] \tag{1.8}$$

Obviously $q_1(s, \xi) \neq q_1(\xi, s)$.

The asymmetry of the function $q_1(s, \xi)$ reflects the properties of the bending moments in an arch with rigidly fixed ends. The bending moments tend toward zero when $\xi \to 0$, but they do not tend toward zero when $s \to 0$, because reactive moments are acting at the points A and B of the arch (Fig. 1). The complete auxiliary radial displacement of a point of the arch will be denoted by $v(s, \xi)$:

$$v(s, \xi) = y(s, \xi) + y_1(s, \xi) \tag{1.9}$$

The complete additional auxiliary load will be denoted by $K(s, \xi)$:

$$K(s, \xi) = q(s, \xi) + q_1(s, \xi) \tag{1.10}$$

Again using the work reciprocity theorem, we arrive at an integral equation of the Fredholm type of the second kind with asymmetrical kernel:

$$u(\xi, \eta) = v(\eta, \xi) - \int_0^l K(s, \xi)\, u(s, \eta)\, ds \tag{1.11}$$

From the properties of the function $v(\eta, \xi)$ and of the kernel $K(s, \xi)$ it can at once be concluded that the function $u(\xi, \eta)$, which satisfies the equation (1.11), also fulfils the boundary conditions of the problem, i.e. the conditions at the rigidly fixed ends of the arch. The procedure of solution of the equation (1.11) does not involve any difficulties [7] and is therefore of no fundamental interest. We call attention to only one property of integral equations derived from the work reciprocity theorem.

As already stated above, the additional auxiliary displacements $y_1(s, \xi)$ must be continuous with respect to s and have derivatives continuous with respect to s up to the fourth order inclusive. Otherwise these displacements are arbitrary. Taking $y_1(s, \xi)$ in the form of a polynomial in terms of s not lower than of the seventh degree, we can always fulfil the following eight conditions:

$$v(0, \xi) = v(l, \xi) = 0, \qquad \frac{\partial v(0, \xi)}{\partial s} = \frac{\partial v(l, \xi)}{\partial s} = 0$$
$$\frac{\partial^2 v(0, \xi)}{\partial s^2} = \frac{\partial^2 v(l, \xi)}{\partial s^2} = 0, \qquad \frac{\partial^3 v(0, \xi)}{\partial s^3} = \frac{\partial^3 v(l, \xi)}{\partial s^3} = 0$$

Under these conditions the work reciprocity theorem leads to an equation of the form (1.11), and this form remains the same for all fixity cases of the arch ends. This latter equation no longer has a unique solution, since it must be satisfied by all forms of arch deflections according to the various boundary conditions. Evidently, in this case the conditions of the Fredholm third theorem, which apply to cases of non-existence of unique solutions of non-homogeneous integral equations [7], must be fulfilled. Thus in this case we must act as follows: having transformed the integrodifferential equations which follow from the work reciprocity theorem into integral equations, it is necessary to examine the solution obtained with respect to fulfillment of the boundary conditions of the problem; if necessary, the general solution must be derived in accordance with the Fredholm theory.

2. Integrodifferential Equations of Equilibrium of Thin Elastic Shells. The example studied in Section 1 fundamentally reflects the essence of the method, the subject of the present paper, of deriving integrodifferential and integral equations of equilibrium of thin elastic shells. The analog of the centroid line of the arch in the theory of shells is the middle surface of the shell, the analog of the centroid line of the beam is the middle surface of the plate - the "map" of the middle surface of the shell. Assume that the internal coordinates of the middle surface of the shell are isothermal coordinates [3]. In this case a line element of the middle surface is defined by the equation

$$ds^2 = F^2(x^1, x^2)[(dx^1)^2 + (dx^2)^2] \tag{2.1}$$

where x^1 and x^2 are the internal coordinates of points of the middle surface of the shell, while $F(x^1, x^2) = F(M)$ is a scalar function of the point $M(x^1, x^2)$ of the middle surface of the shell. We assume that the coordinates x^i are also Cartesian coordinates of the middle plane of the plate.

This establishes a one-to-one correspondence of the points of the middle surface of the shell to those of the middle plane of the plate. Formula (2.1) states a relation between the line element ds of the middle surface of the shell and the line element ds_0 of the middle plane of the plate. This relation is invariant. In an arbitrary system of orthogonal coordinates in the middle plane of the plate we get*

$$ds^2 = F^2(M)\, ds_0{}^2 = F^2(M)\, G_{ii}(M)\, (dx^i)^2 \tag{2.2.1}$$

where the G_{ii} are the components of the metric tensor in the middle plane of the plate. In the middle surface of the shell the components $g_{ik}(i, k = 1, 2)$ of the metric tensor are

$$g_{ii} = F^2(M)\, G_{ii}(M), \qquad g_{ik} = 0 \qquad (i \neq k) \tag{2.22}$$

Assume a system of coordinate lines $x^3 = z$ in the plate and in the shell, coinciding with the normals to their respective middle surfaces. The vectors of the coordinate base shall be denoted by $\mathbf{e}_i$. The modulus $|\mathbf{e}_3| = 1$. The components of the metric tensor g_{i3} are zero when $i \neq 3$, while $g_{33} = 1$. The coordinates $(i = 1, 2, 3)$ arithmetize the spaces within the shell and plate.

Assume the thickness $2h$ of the shell to equal that of the plate; we further assume for reasons of simplification that h is a constant, although the method developed below can easily be generalized to include shells of variable thickness. Finally we assume that the elastic constants of the material are the same for shell and plate. Subsequently we will use the Kirchhoff-Love hypothesis concerning "invariable straight line normals" to the middle surface.

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* Here and subsequently we use the summation symbol known from tensor analysis.

We turn to the derivation of the integrodifferential equations of equilibrium of shells. By $v_{(a)\,i}(N, M)$ let us denote the displacement of a point N of the middle plane of the plate in the direction of the coordinate line i produced by the action of a concentrated unit force applied at the point M of the middle plane and directed along the coordinate line a. We know that

$$v_{(\alpha)i}(N, M) = v_{(i)\alpha}(M, N) \tag{2.3}$$

We will consider the functions $v_{(a)\,i}(N, M)$ as covariant components of the displacement vector of the points of the middle surface of the shell. The system of forces producing the displacements $v_{(a)\,i}(N, M)$ in the shell consists of some applied concentrated force acting at a point M, an applied load distributed along the middle surface of the shell, and the reactions of the constraints.

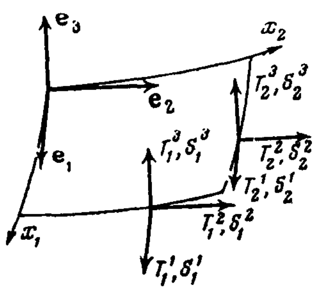

Fig. 3.

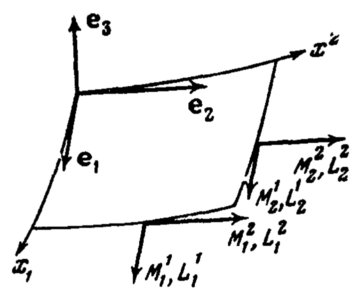

Fig. 4.

The concentrated force corresponding to the displacements $v_{(a)\,i}(N, M)$ in the shell will be determined by applying Hooke's law when $a = 1, 2$, and by the equilibrium equations of the shell in terms of displacements when $a = 3$. In this way it is possible to arrive at the following conclusion: a concentrated unit force applied to the middle plane of the plate transforms into a concentrated force acting on the middle surface of the shell. The vector of the transformed force is approximately determined by the contravariant components*

$$Y^i_{(k)} = \begin{cases} F^{-2}(G_{ii})^{-1/2} & (i = k), \\ 0 & (i \neq k), \end{cases} \qquad g_{33} = G_{33} = 1 \tag{2.4}$$

The displacements $v_{(a)\,i}$ in the shell are produced by the concentrated forces obtained above and the distributed load acting on the boundary

* In addition to the terms indicated in (2.4) the exact expression for the components of the transformed force contains those of the order $h^2 k_1{}^2$ where k_1 is the principal curvature of the shell. It is known that these terms can be neglected in comparison with unity [1].

surfaces of the shell. On the basis of the hypothesis of non-deformable normals to the middle surface the load acting on the surface of the shell can be replaced by a statically equivalent system of forces and moments acting on the middle surface of the shell. By $K_{(\alpha)}{}^{j}$ and $H_{(\alpha)}{}^{j}$ we denote the contravariant components of the intensities of the applied loadings by forces and moments corresponding to the displacements $v_{(\alpha)i}$ and referred to the middle surface of the shell. Further, by $S_{(\alpha)k}{}^{j}$ and $L_{(\alpha)k}{}^{j}$ we denote the contravariant components of the stress resultants and moments, acting on the boundary line of an element of the middle surface of the shell. The direction of these stress resultants and moments, as well as the meaning of the used notations are shown in Figs. 3 and 4.

As everywhere, the subscript given in parentheses and not appearing in Figs. 3 and 4 characterizes the direction of the concentrated applied force which produces the auxiliary displacement. If the boundary of the middle surface is not represented by coordinate lines, then the stress resultants and moments acting along this boundary line will be denoted by $S_{(\alpha)}{}^{j}$ and $L_{(\alpha)}{}^{j}$.

The load corresponding to the displacements $v_{(\alpha)i}$ is to be determined as follows:

1. If $\alpha = 1, 2$, the load is determined by means of Hooke's law, which permits of finding the stresses on the boundary surfaces of the shell and then reducing them to the middle surface. In this manner it is possible to avoid the appearance of improper divergent integrals in the integrodifferential equations of equilibrium of shells [4c]. Considerable simplifications can be achieved here by application of the Kirchhoff-Love hypothesis [8].

2. If $\alpha = 3$, we use the elastostatic system of equations of the theory of shells [4a - 4c] replacing equation (1.3). In applying the work reciprocity theorem the system of displacements $v_{(\alpha)i}(N, M)$ and the corresponding loading of the middle surface will be considered as a system of auxiliary displacements and forces.

By $u_{(i)\alpha}(M, N)$ we denote the covariant component of the displacement vector of the point M of the middle surface of the shell, this displacement being produced by the action of the applied unit force at the point N directed along the coordinate line i. The contravariant components of this unit force are given by the formulas

$$X_{(i)}{}^{k} = \begin{cases} F^{-1}\,(G_{ii})^{-1/2} & (k = i = 1,2), \\ 0 & (k \neq i), \end{cases} \quad X_{(3)}{}^{k} = \begin{cases} 1 & (k = 3) \\ 0 & (k \neq 3) \end{cases} \tag{2.5}$$

The stress resultants and moments corresponding to the displacements

$u_{(i)\alpha}$ and acting on the boundary line of an element of the middle surface will be denoted by $T_{(i)j}^{k}$ and $M_{(i)j}^{k}$. The mutual position of the vectors of these stress resultants and moments, and the meaning of the notations, are illustrated in Figs. 3 and 4. The subscript in parentheses, not indicated in Figs. 3 and 4, always characterizes the direction of the concentrated force producing the basic displacements. By analogy with preceding statements, we will denote by $T_{(i)}^{k}$ and $M_{(i)}^{k}$ stress resultants and moments acting on such parts of the boundary line of the middle surface of the shell as do not coincide with coordinate lines. The stress resultants and moments acting along the boundary line of the middle surface are the reactions of the supports of the shell. The system of displacements $u_{(i)\alpha}$ and of the forces corresponding to these displacements will be considered as the fundamental system in applying the work reciprocity theorem. With all this in view, and by virtue of the theorem of the work reciprocity theorem as applied to the fundamental and auxiliary systems of displacements and forces, we get

$$\begin{aligned} u_{(i)\alpha}(M, N) = {} & F^2(M)\sqrt{G_{\alpha\alpha}(M)}\, X_{(i)}{}^{k}(N)\, v_{(\alpha)k}(N, M) - \\ & - \iint\limits_{(S)} F^2(M) + \sqrt{G_{\alpha\alpha}(M)}\, [K_{(\alpha)}{}^{j}(Q, M)\, u_{(i)j}(Q, N) + \\ & + H_{(\alpha)}{}^{j}(Q, M)\, \omega_{(i)j}(Q, N)]\, dS_Q + \\ & + \oint F^2(M)\sqrt{G_{\alpha\alpha}(M)}\, [T_{(i)}{}^{j}(Q, N)\, v_{(\alpha)j}(Q, M) + M_{(i)}{}^{j}(Q, N)\, \psi_{(\alpha)j}(Q, M) - \\ & - S_{(\alpha)}{}^{j}(Q, M)\, u_{(i)j}(Q, N) - L_{(\alpha)}{}^{j}(Q, M)\, \omega_{(i)j}(Q, N)]\, ds_Q \end{aligned} \tag{2.6}$$

where the area integral extends over the middle surface of the shell, while the line integral is taken over the boundary line of the area mentioned. The fundamental and the auxiliary bending moments acting along the boundary line of the middle surface are here denoted by $M_{(i)}{}^{j}$ and $L_{(\alpha)}{}^{j}$ respectively. The terms depending on the twisting moments are included into the fundamental and into the auxiliary stress resultants $T_{(i)}{}^{j}$ and $S_{(\alpha)}{}^{j}$ respectively [5].

The rotation angles at their footpoints of the normals to the middle surface, corresponding to the fundamental and the auxiliary systems of displacements, are denoted by $\omega_{(i)j}$ and $\psi_{(\alpha)j}$.

If we assume that the coordinate lines x^i of the middle surface coincide with its curvature lines, and if we use the hypothesis of nondeformability of the normals, we then arrive at the formulas

$$\omega_{(i)1} = \frac{1}{2F^2}(\partial_2 u_{(i)3} - \partial_3 u_{(i)2}) = \frac{1}{F^2}(\partial_2 u_{(i)3} + k_2 u_{(i)2}) \tag{2.7.1}$$

$$\omega_{(i)2} = \frac{1}{2F^2}(\partial_3 u_{(i)1} - \partial_1 u_{(i)3}) = -\frac{1}{F^2}(\partial_1 u_{(i)3} + k_2 u_{(i)1}) \qquad (2.7.2)$$

Here and subsequently we use the notations $\partial/\partial x^i = \partial_i$, while the k_i represent the principal curvatures of the middle surface of the shell.

Since $H_{(a)}{}^3 \approx 0$, $\omega_{(i)3}$ does not appear in equation (2.6). The rotation angles $\psi_{(a)j}$ are to be determined analogously to (2.7.1) and (2.7.2).

Equations (2.6) represent the system of integrodifferential equations of equilibrium corresponding to the linear theory of shells; this system of equations permits of determining the components $u_{(i)a}$ of Green's tensor (influence function). Equations (2.6) are an analog of equations (1.5). Integration by parts will make it possible to eliminate the derivatives of the displacements $u_{(i)j}$ from the integrand of the double integral. Taking advantage of the arbitrariness in the choice of the regular part of the auxiliary displacements $v_{(a)i}$, it is also possible to simplify the curvilinear integral and in some cases even to eliminate it [4a, c]. This transformation of equations (2.6) is analogous to reducing equation (1.5) to the form (1.11). The influence functions determined from equations (2.6) permit of finding the displacements produced by an arbitrary loading of the middle surface of the shell.

The further development of the method indicated will be achieved by working out a particular concrete example.

3. Example, Spherical Dome. Equations analogous to (2.6) have been applied to cases characterized by the condition

$$F(M) \equiv 1 \qquad (3.1)$$

Equilibrium equations of cylindrical [4-8] and shallow [9] shells have been considered. Condition (3.1) finds approximate fulfilment in the case of shallow shells [1]. As a very simple example we will study the equilibrium equations of a spherical dome of radius R. This example of the case $F(M) \neq 1$ will also permit of clarifying some general features of the method.

Assume the origin of the Cartesian system of coordinates to coincide with one of the poles of the sphere and the axis Oz to be a diameter of the sphere. Consider the plane Q to be tangent to the surface of the sphere at the point $N(0, 0, 2R)$. The stereographic projection of the surface of the sphere on this plane can be given parametrically by introducing the spherical coordinates θ, ϕ on the surface of the sphere:

$$x = R\sin\theta\cos\varphi, \qquad y = R\sin\theta\sin\varphi, \qquad z = R + R\cos\theta = 2R\cos^2\frac{1}{2}\theta \qquad (3.2)$$

$$x^* = 2R\ \tan\frac{1}{2}\theta\cos\varphi, \qquad y^* = 2R\,\mathrm{tg}\,\frac{1}{2}\theta\sin\varphi, \qquad z^* = 2R \qquad (3.3)$$

where $M(x, y, z)$ is a point on the sphere and $M^*(x^*, y^* z^*)$ its stereographic projection on the plane Q. Equations (3.2) and (3.3) show that the part $0 \leqslant \theta \leqslant \theta_0$ becomes mapped on a circle of radius $2R \tan 1/2\, \theta_0$ in the plane Q. The auxiliary system of displacements is represented by the displacements of a round plate. The line elements ds and ds_0 on the surface of the sphere and on its stereographic projection are given by the formulas

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad ds_0{}^2 = \frac{R^2}{\cos^4 {}^1\!/_2\theta}(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{3.4}$$

respectively, from which

$$ds^2 = \cos^4 \frac{1}{2}\,\theta\, ds_0{}^2 \tag{3.5}$$

Thus x^* and y^* are isothermal coordinates. We will not use the coordinates x^* and y^*, keeping the coordinates $x^1 = \theta$ and $x^2 = \phi$ instead. From (2.2.1) and (3.5) we find

$$F(M) = \cos^2 \frac{1}{2}\,\theta, \qquad G_{11} = G_{\theta\theta} = R^2\left(\cos\frac{1}{2}\theta\right)^{-4}, \qquad G_{22} = G_{\varphi\varphi} = R^2\sin^2\theta\left(\cos\frac{1}{2}\,\theta\right)^{-4} \tag{3.6}$$

To establish the system of the auxiliary displacements $v_{(\alpha)j}$ we will use the known solutions of the problem of equilibrium of a round plate acted upon by a concentrated force [5, 6]. These solutions satisfy the conditions

$$v_{(\alpha)3}(P, M) = v_{(3)\alpha}(P, M) = 0 \quad (\alpha = 1{,}2) \tag{3.7.1}$$

where P is an arbitrary point on the middle plane of the plate. Along the circular boundary line C in that plane the conditions

$$v_{(3)3}(Q, M) = 0, \qquad \frac{\partial v_{(3)3}(Q, M)}{\partial n} = 0 \tag{3.7.2}$$

are fulfilled [5]. Q is a point on the boundary line, while n is a normal to C.

Now refer to the statements in Section 2 on determining the loading corresponding to the displacements $v_{(\alpha)j}$ of the middle surface of the shell. To simplify equations (2.6) when $\alpha = 1, 2$, we use the particular variation of the Kirchhoff-Love hypothesis according to which a prismatic element of the shell with generators normal to the middle surface of the latter is in the state of plane stress [1]*. Then

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* Another version of the Kirchhoff-Love hypothesis used to be applied [8,9]. In the application of the hypothesis just mentioned we take advantage of the arbitrary character of the distribution of the auxiliary displacements along the thickness of the shell [4a] by establishing a system of displacements $v_{(\alpha)j}$ which fulfil the conditions of the hypothesis rigorously. In doing so we disregard terms of the order $h^2 k_i{}^2$ in the expressions for the components of the transformed concentrated force.

$$K_{(\alpha)}{}^{j} = 0, \qquad H_{(\alpha)}{}^{j} = 0 \qquad (\alpha = 1,\ 2;\ j = 1,2,3) \tag{3.8}$$

The kernels $K_{(3)}{}^{j}$ will be found from the elastostatic system of equations of the shell theory in a manner analogous to that used in determining the kernel of the equation (1.5). On the basis of the Kirchhoff-Love hypothesis we assume the kernels $H_{(3)}{}^{j}$ to be zero. Let the spherical dome be supported along a circle of latitude $\theta = \theta_0$ with the boundary of the dome rigidly fixed in accordance with the conditions

$$u_{(i)\alpha}(Q, N) = 0, \qquad \omega_{(i)2}(Q, N) = 0 \qquad (i,\ \alpha = 1,\ 2,\ 3) \tag{3.9}$$

The points $M(\theta, \phi)$, $N(\theta_1, \phi_1)$ and $P(\theta', \phi')$ are situated on the middle surface of the shell. The point $Q(\theta_0{}', \phi_0{}')$ is located on the boundary line of the middle surface. Equations (1.6) assume the following form:

$$u_{(i)\alpha}(M, N) = R \sin^{\alpha-1}\theta \cos^2 \tfrac{1}{2}\theta \left\{ v_{(\alpha)[i]}(N, M) + R \sin\theta_0 \times \right.$$
$$\left. \times \int_0^{2\pi} [T_{(i)}{}^{j}(Q, N)\, v_{(\alpha)j}(Q, M) + M_{(i)}{}^{j}(Q, N)\, \psi_{(\alpha)j}(Q, M)]\, d\varphi' \right\} \tag{3.10.1}$$

$$u_{(i)3}(M, N) = \cos^4 \tfrac{1}{2}\theta \left\{ v_{(3)i}(M, N) - \iint_{(S)} K_{(3)}{}^{j}(P, M)\, u_{(i)j}(P, N)\, dS_p \right\} \tag{3.10.2}$$
$$(i, j = 1,\ 2,\ 3;\ \alpha = 1,\ 2)$$

Here $v_{(\alpha)[i]}$ is the projection of the displacement $v_{(\alpha)}$ on the direction of the coordinate line i. Equations (3.10.1) and (3.10.2) represent a system of the integrodifferential equations of equilibrium of a spherical dome. This system is analogous to equation (1.5). The solution of the system (3.10.1) and (3.10.2) can be reduced to the solution of one integral equation of Fredholm type of the second kind. To this end it is sufficient to make use of the arbitrary choice of the regular part of the displacements $v_{(\alpha)j}$. Noting that the functions $v_{(\alpha)j}(Q, M)$ are regular for all values ϕ', instead of the displacements $v_{(\alpha)j}$ we introduce into the equations (3.10.1) the displacements

$$V_{(\alpha)j}(P, M) = v_{(\alpha)j}(P, M) - v_{(\alpha)j}(Q, M) \qquad (\alpha = 1,2) \tag{3.11}$$

Then we find from (3.10.1)

$$u_{(i)\alpha}(M, N) = R \sin^{\alpha-1}\theta \cos^2 \tfrac{1}{2}\theta V_{(\alpha)[i]}(N, M) \tag{3.12}$$

It is immediately clear that the functions (3.12) satisfy the boundary conditions (3.9). By virtue of (3.12) equation (3.10.2) assumes the form

$$u_{(i)3}(M,N)=\Phi_{(i)3}(M,N)-\iint\limits_{(S)}K_{(3)}^{3}(P,M)\,u_{(i)3}(P,M)\,dS_p \tag{3.13}$$

where $\Phi_{(i)3}$ is a known function. The solution of the problem of elastic equilibrium of a spherical dome is reduced to the solution of the integral equation (3.12). This solution can be obtained by means of known methods and need not be discussed here.

4. Questions of Equivalence and Uniqueness. The validity of the work reciprocity theorem in the theory of shells and the compatibility of this theorem with the Kirchhoff-Love hypothesis have been rigorously proved [2]. Therefore the accuracy of the results obtained above lies within the general accuracy limits of the theory of equilibrium of thin shells. In this sense equations (2.6) and their consequences are equivalent to the differential equations of equilibrium of shells based upon the Kirchhoff-Love hypothesis.

The integrodifferential equations (2.6) have no unique solution. The integral equations following from equations (2.6), e.g. equation (3.13), can have a unique solution, which in this case is that required. The possibility has, however, been emphasized (see Section 1) that the integral equations of equilibrium derived from the integrodifferential equations may have no unique solution at all, which is in conformity with the third theorem of Fredholm. Therefore the integral equations obtained in the manner indicated must undergo an additional examination, or a method of solution known to secure fulfilment of the boundary conditions must be used.

In conclusion we note that the above method permits of developing effective numerical procedures for the solution of boundary value problems of the theory of shells.

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